

Gödel Incompleteness and the Black Hole Information Paradox

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Semiclassical reasoning suggests that the process by which an object collapses into a black hole and then evaporates by emitting Hawking radiation may destroy information, a problem often referred to as the black hole information paradox. Further, there seems to be no unique prediction of where the information about the collapsing body is localized. We propose that the latter aspect of the paradox may be a manifestation of an inconsistent self-reference in the semiclassical theory of black hole evolution. This suggests the inadequacy of the semiclassical approach or, at worst, that standard quantum mechanics and general relativity are fundamentally incompatible. One option for the resolution for the paradox in the localization is to identify the Gödel-like incompleteness that corresponds to an imposition of consistency, and introduce possibly new physics that supplies this incompleteness. Another option is to modify the theory in such a way as to prohibit self-reference. We discuss various possible scenarios to implement these options, including eternally collapsing objects, black hole remnants, black hole final states, and simple variants of semiclassical quantum gravity.

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I. INTRODUCTION

Attempts thus far to combine general relativity and quantum mechanics, the two cornerstones of our description of nature, have led to difficulties, an example of which is the black hole information paradox (BHIP) [1, 2]. BHIP suggests that the process by which an object collapses into a black hole and then evaporates by emitting Hawking radiation is not unitary. The effect apparently leads to a new kind of unpredictability, quite apart from the conventional one associated with Heisenberg uncertainty. The derivation of the paradox employs a semiclassical treatment of quantum fields localized close to the event horizon of a black hole, which would seem to leave open the possibility of resolution through a more detailed treatment of quantum gravity. However, as the problem can be posed of a region near the horizon of a large black hole, it need not invoke a strong gravitational field, which suggests that the problem is amenable to a local quantum field theoretic treatment [3]. On the other hand, from the string theory standpoint it has been argued that the detailed knowledge of the Planck-scale physics cannot be ignored even if there is no strong curvature or other coordinate-invariant manifestation of the event horizon. Arguably, the issue is still open, and continues to attract efforts at resolving [2, 3, 4, 5, 6, 7] and clarifying it [8]. In particular, in the loop quantum gravity approach, quantum effects eliminate black hole singularities. As a result, one can in principle track information to the future of a would-be singularity [9], thereby preserving information. It has also been argued that BHIP may be avoided by attributing Hawking radiation solely to quantum decoherence, considering that pure states remain pure under unitary, closed-system evolution [10, 11]. This is consistent with the viewpoint that pure quantum states do not form black holes [12].

In the present work, we propose that BHIP, in particular the question of localization of information in an evaporating black hole, may be indicative of an inconsistent self-reference occurring in the semiclassical treatment of black hole evolution [13]. Admittedly, a rigorous study of this claim would require an axiomatization of the semiclassical theory. Nevertheless, we believe there are plausible grounds for believing that there are features, presented here, that any such axiomatic theory should satisfy. Inspite of the very abstract nature of this approach to black hole evolution, we will be led below to concrete, nontrivial consequences for black hole formation. This work may be primarily regarded as a plea for injecting metamathematical considerations in the study of fundamental physics such as quantum gravity, and BHIP in particular. A similar case can be made for applying quantum information theoretic and computation theoretic insights to understanding the basic mathematical structure of physical laws [15].

The remaining article is arranged as follows. In Sections IA and IB, we briefly review the black hole information paradox and Gödel's first incompleteness theorem, respectively. The ambiguity in the localization of information

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falling into an evaporating black hole is introduced in Section II. An argument that the localization problem may be a formal inconsistency in the semiclassical theory is presented in Section II A, and that this inconsistency can be viewed as self-referential in origin is presented in Section II B. The question of restoring consistency by invoking various ways to account for the Gödel incompleteness obtained by imposing consistency is considered in Section III. That of restoring consistency by means of avoiding self-reference is considered in Section IV. Finally, we conclude in Section V.

A. The black hole information paradox

A brief introduction to BHIP is as follows. We denote by H_M the Hilbert space of a collapsing body M , of dimension N , where $N = e^S$, and S is the black hole's entropy. In the semiclassical treatment of quantum field fluctuations on the background spacetime determined by the collapse and evaporation of a black hole, the Hilbert space of the fluctuations can be separated into two subsystems, given by Hilbert spaces, respectively, H_{in} and H_{out} (each also of dimension N), located inside and outside the horizon. The correlations between the two fields is characterized by the Unruh quantum state $|\Phi\rangle_{\text{in+out}}$, which looks like the vacuum in the far past, a maximally entangled pure state [7]

$$|\Phi\rangle_{\text{in+out}} = \frac{1}{\sqrt{N}} \sum_{j=1}^N |j\rangle_{\text{in}} |j\rangle_{\text{out}}, \quad (1)$$

where $|j\rangle_{\text{in}}$ and $|j\rangle_{\text{out}}$ are orthonormal bases for H_{in} and H_{out} , respectively. The Unruh state contains a flux of particles in H_{out} , that constitutes the Hawking radiation.

To the outside observer H_{in} is inaccessible, and the field localized outside is in the maximally mixed state $\sum_j (1/N) |j\rangle_{\text{out}} \langle j|_{\text{out}}$ containing no detailed information about M . When back-reaction is included in the semiclassical approximation, the black hole will slowly lose its mass through Hawking radiation, and disappear. From the classical black hole geometry, the information about what formed the black hole cannot come out without violating causality. So at late times, we obtain a mixed state, even though M began in a pure state. Clearly, this process cannot be described as a unitary evolution, and suggests that black holes destroy information. This problem is often referred to as BHIP. However, it is convenient for us to regard it as one aspect of the full paradox, which (aspect) we shall call the black hole information loss problem. There is another aspect of the paradox, introduced in Section II, which we call the black hole information localization problem.

B. Gödel Incompleteness

A formalization or axiomatization of arithmetic (in general, any deductive theory) is the reduction of arithmetic to a small set of initial formulas and rules of symbolic manipulation, such that a chain of formulas obtained by manipulation in the formal system corresponds to and represents deductions in arithmetic. By looking at the correspondence between the formal system and the deductive theory in reverse, Hilbert originated metamathematics, his name for the study of rigorous proof in mathematics and symbolic logic. Here a formal system is a ‘game’ constructed, independently of its interpretation, as a sequence of formulas obtained mechanically according to the rules of symbolic manipulation, starting from the initial formulas. The formal system is interpreted as representing the deductive system if the initial formulas can be interpreted as expressing the axioms of the theory, and the rules of symbolic manipulation, its logical rules of inference. Then, a metamathematical proof that a formula occurs in a sequence of formulas of the formal system yields a proof that the proposition which is the interpretation of this formula is a theorem of the deductive theory.

From the standpoint of mathematic logic, it is important to distinguish between statements in a deductive theory from meta-statements in the metatheory, which studies concepts, proofs and truths in the theory. Failure to do so can lead to inconsistency through self-reference, of which a well-known example is the liar's paradox: “this statement is false”. Here the statement acts as its own metastatement. If the statement is true, then it is false, and conversely: a contradiction. From a syntactic viewpoint, a formal system is consistent if for any proposition α , at most one of α and its negation $\neg\alpha$ is provable. The formal system is complete if for any proposition α , at least one of α and $\neg\alpha$ is provable [16].

Gödel's (first) incompleteness theorem, perhaps the most celebrated result in metamathematics, states that any formal system that is (1) rich enough to encompass arithmetic, (2) is finitely specified, and (3) consistent, contains a proposition that can neither be proved nor refuted within the system [18], and is thus incomplete. Here ‘finitely specified’ means that there is an algorithm to list all axioms (initial formulas) and rules of inference (rules for symbolic manipulation), which may be countably infinite. Regarding (3), Gödel actually requires the stronger condition

of ω -consistency [19], a subtlety we may ignore here. Every deductive theory that includes elementary arithmetic (the notions of natural numbers, and of the operations of addition and multiplication) also inherits this incompleteness. Only theories with sufficiently simple logical structure, such as propositional or sentential calculus, Presburger arithmetic and elementary geometry, are complete.

Gödel's theorem is a consequence of the fact that arithmetic has enough expressive power to allow meta-arithmetic statements to be mirrored into it, thus making some sort of self-reference unavoidable. Crucial to Gödel's proof is the observation that the symbols of a formal arithmetic system, and hence formulas and proofs constructed in it, can be assigned a unique number, now called the Gödel number. Any other method of assigning numbers to these objects in one-to-one fashion will also work. As a result, meta-arithmetical statements about arithmetic can be paraphrased arithmetically as statements about their Gödel numbers. The meta-arithmetic statement that a sequence α of formulas is a proof of the formula β , or that formula γ is provable, can be expressed, respectively, as an arithmetical relation between the Gödel numbers for α and β , or an arithmetic property of the Gödel number of γ , and thus expressed in the formal system. This isomorphic mapping of meta-arithmetic into arithmetic opens up the danger of self-reference. If one takes care to set up blocks that prohibit inconsistency of the liar's paradox type, one is left with incompleteness as a side-effect, as it were.

Let us briefly present a simplified, illustrative but unrigorous sketch of Gödel's proof. Let system P be a formalization of ordinary arithmetic, whose alphabet \mathbf{P} consists of the symbols "0" (zero), "s" (successor), the logical constants \forall , \neg (negation), \vee , and variables of the first type x_1, y_1, \dots (for individuals, the numbers including 0), variables of the second type x_2, y_2, \dots (for classes of individuals), and so on. Metamathematically, it is immaterial what symbols we choose to represent these basic signs, and we may choose natural numbers for them. Accordingly, a formula is a finite series of natural numbers, and a particular proof is a finite series of a finite series of natural numbers. Metamathematical concepts and propositions thereby become concepts and propositions concerning natural numbers, and therefore, at least partially expressible in the symbols of P itself. In particular, Gödel shows that the metamathematical concepts "formula", "axiom", "variable", "proof-schema", "provable formula", etc., are definable within P .

We call formulas involving a single variable as class-signs. If α is a class-sign, and t a number, we designate by $[\alpha; t]$ the formula obtained by substituting the sign for t in place of the free variable in α . Let every class-sign be somehow ordered, e.g., lexicographically. The concept class-sign and ordering R can be defined in P . Let $R(n)$ denote the n th class-sign.

We define a set K of whole numbers by:

$$n \in K \equiv [R(n); n] \text{ is not provable in } P. \quad (2)$$

As the r.h.s of Eq. (2) is definable in P , so is the concept K in the l.h.s. That is, there is a class-sign W in P such that $[W; n] \equiv n \in K$. For some positive integer q , $W = R(q)$. We will find that the string, a Gödel sentence for P ,

$$[R(q); q], \quad (3)$$

is *undecidable* in P . If proposition (3) were provable in P , then so would the proposition $q \in K$ by definition. The latter would imply $[R(q); q]$ is *not* provable in P according to Eq. (2). This is a contradiction. On the other hand, if proposition (3) were refutable in P , i.e., $\neg[R(q); q]$ were provable in P , this would supply the proof that $\neg(q \in K)$, so that, by Eq. (2), $[R(q); q]$ is provable in P . Again, we obtain a contradiction. Therefore, assuming P is consistent, $[R(q); q]$ is undecidable in P [20]. Thus P is incomplete. Proposition (3), which involves supplying a formula its own serial number as its argument, is an instance of the diagonal argument [22], pioneered by the mathematician Cantor [23]. Clearly, (3) is true, since if it were false, it would be provable in P , thereby contradicting itself. We thus have the curious situation that (3) is known to be true metamathematically, even though it is unprovable in P [24].

An existential proof of Gödel's theorem is obtained by noting that the set Π of provable propositions in a formalization of arithmetic is recursively enumerable (r.e.) [25], in fact, recursive [26], whereas the set T of truths expressible in arithmetic is not r.e [17]. In a (semantically) consistent formalization, clearly, $\Pi \subseteq T$. Since T is not r.e, there should be truths that are unprovable in the given formalization.

Gödel's incompleteness theorem is related to Turing uncomputability, the unsolvability of certain problems algorithmically [23, 27]. If every proposition in an arithmetic system P were decidable by an algorithm G , this could serve as the basis to solve the halting problem for Turing machines, which is known to be undecidable [28]. The unsolvability of the halting problem thus implies the existence of undecidable propositions in P .

Turing machines and the system P derive their power of self-reference from their universality: the existence of universal Turing machines in the case of the former, and the ability to mirror meta-arithmetic statements in the case of the latter. However we will find that one can construct simpler systems in which self-reference occurs, leading to incompleteness or inconsistency. As an informal example, we note disentangling proposition (3) that it asserts its own unprovability in P . It may thus be regarded as the consistent, and hence incomplete, version of the liar's paradox, which, by asserting its own falsehood, is complete, but inconsistent.

II. BLACK HOLE INFORMATION LOCALIZATION PROBLEM

A problem in BHIP closely related to the information loss problem concerns the situation that as a radiating semiclassical Schwarzschild black hole shrinks, and possibly fully evaporates, there seems to be no unique prediction of where the information about M is localized. The event horizon being a globally defined property of a spacetime, to a freely falling observer (called Alice), matter falling towards a black hole encounters nothing unusual while crossing the horizon. She finds the quantum information contained in the initial matter M pass freely into the interior of the black hole. In contrast, according to an observer Bob, who is outside the event horizon, at a large distance from the black hole and approximately at rest with respect to it, the collapsing or infalling object appears to increasingly slow down and to freeze asymptotically at the horizon. He finds that it never quite crosses the event horizon during the semiclassical stage, and perhaps even later, as the black hole evaporates, possibly entirely. This lack of a unique prediction of the localization of the infalling/collapsing matter in the semiclassical theory is the black hole information localization problem. This can be made clearer as follows.

A. The black hole information localization problem viewed as formal inconsistency

We consider Alice falling towards a Schwarzschild black hole of mass $m > 0$. To Bob, the coordinate time t of the Schwarzschild metric corresponds approximately to his proper time. Alice initially stands close to Bob before propelling herself forward and then allowing herself to fall freely into the black hole. The black hole mass m is assumed to be sufficiently large that, from her viewpoint, all events in her worldline segment up to her infall into the singularity can be regarded to good accuracy as happening in a region of spacetime endowed with a classical, time-independent metric. Let ϵ denote the event of this worldline of Alice intersecting the horizon. Consider the Kretschmann scalar $K^A(\tau) = R_{abcd}(\tau)R^{abcd}(\tau)$, where $R_{abcd}(\tau)$ is the Riemann tensor along Alice's worldline, parametrized by her proper time τ . For convenience, we set $\tau = 0$ at ϵ . For a Schwarzschild metric (4), $K^A = 48m^2/r^6$ [29]. In particular, event ϵ is marked by the scalar value $K^A(0) = 3/4m^4$.

According to Bob, because of gravitational redshift, Alice is falling ever more slowly, but never quite getting to the horizon. Throughout the semiclassical regime, we may assume that the evaporating black hole is approximately static and spherically symmetric, with a the shrinking horizon, and that Bob remains the distant, stationary observer. Thus, even when substantial black hole mass has evaporated, Alice will not yet have crossed through the horizon, as viewed from his perspective. A quantum gravity scenario in which she never does so as seen by Bob is not inconceivable. Indeed, this is the accepted situation in string theory [3]. Even if Alice's crossing the horizon does eventually happen, this event (denoted η) would presumably have to occur in the strongly quantum gravity regime. We expect that the Kretschmann scalar $K^B(0)$ at η would be far larger. Even if $K^A(0) = K^B(0)$ by an extraordinary conspiracy, the functions $K^A(\tau)$ and $K^B(\tau)$ could not be the same for $\tau \geq 0$, as the former occurs in a purely classical spacetime, whereas the latter in a full quantum gravity regime. In a scenario where Alice does not cross the horizon in Bob's perspective, η is termed a "null event".

Either way, we are led to conclude that $\epsilon \neq \eta$, and that Alice's and Bob's perspectives are mutually incompatible. To the extent that the assumptions made are realistic, this inability of the semiclassical theory to assign a unique spacetime location to a physical event [30] may be regarded as a formal inconsistency in the theory. In particular, let \mathbf{g} be the proposition that the quantum information pertaining to an infalling body passes through the horizon at event ϵ . From Alice's perspective, one predicts \mathbf{g} , and from Bob's perspective, one predicts $\neg\mathbf{g}$. The inconsistency is that the theory does not (seem to) assign a unique destiny to the infalling information.

It is worth stressing that this incompatibility is fundamental, and should not be thought of as a manifestation of non-Boolean quantum logic defined in the state space $H_M \otimes H_A \otimes H_B$, where H_A and H_B are the Hilbert spaces of Alice and Bob, respectively. In particular, the incompatibility cannot be accounted for by an entanglement between the black hole, Alice and Bob, where ϵ (η) occurs relative to Alice (Bob). The simple reason for this is that distant Bob need not interact, and thus need not become entangled, with M and Alice. But the basic reason is that such a 'relative state' interpretation is possible only if Alice and Bob are two states of the *same* system, rather than two different systems, as is the case here [31].

In retrospect, the inconsistency stems from the fact that general relativity permits the co-existence of observers of very disparate powers: on the one hand, Bob, and on the other, Alice, who is infinitely more powerful than him in the sense that there are events (such as ϵ) to his infinite future that happen in finite proper time according to Alice (but none vice-versa). In fact, a disparity of this kind may serve as a basis to construct a general relativistic hypercomputer to compute Turing uncomputable functions [32]. The origin of the localization problem, then, seems to be the combined effect of the existence and diminution of the domain of the 'infinitely powerful' observer, given by the interior of the Schwarzschild black hole.

To our knowledge, of the various efforts to resolve BHIP, currently string theory alone seems to acknowledge the

localization problem. According to this view, an infalling object carries its information intact through the horizon as seen in Alice's perspective. Bob perceives the black hole draped by a heated membrane that is the source of the Hawking radiation, and situated just above the event horizon. He considers any infalling information to become disrupted upon approaching this membrane, and re-emitted as radiation to the exterior universe, keeping the late-time state pure. While this offers the prospect of solving the information loss problem, that of information localization still remains.

The standard understanding among string theorists [3] is that this perspective-dependent nonlocal existence of infalling matter can be accounted for by the principle of black hole complementarity [33], the idea that Alice's description is complementary to Bob's, somewhat in the same way that the description of a quantum particle in terms of position and momentum are complementary. Historically, this notion of nonlocality has served to motivate the holographic principle [34]. Thus the incompatibility of Alice's and Bob's perspectives is treated as a new principle of relativity, rather than as an inconsistency.

The idea that the observation or non-observation of an event can depend on the the observer's reference frame occurs also in the context the Unruh effect [35], where the Unruh radiation is observed in an accelerated frame but not in the corresponding inertial frame. Indeed, the existence of Unruh radiation can be linked to the apparent event horizon perceived by an accelerated observer, thus putting it in the same conceptual framework as Hawking radiation [36]. It has long been argued that the appearance of particle events is observer dependent [37]. Thus, the observation or non-observation of an event of horizon-crossing or not crossing can reasonably depend on the frame of the observer and does not necessarily signal inconsistency.

In string theory, this remarkable position is justified by appeal to a verificationist philosophy: one can choose not to be bothered by the ‘cloning’ of information because it cannot be verified to have occurred by an observer within the semiclassical effective theory regime [33], inasmuch as any attempt to do so would require energy far beyond the Planck scale. Thus, this standpoint defers a treatment of the problem from the semiclassical regime to a full theory of quantum gravity.

B. BHIP viewed as an inconsistent self-reference in quantum gravity

Since physics is described in the language of mathematics, it is a deep yet natural question to ask what, if any, is the impact of Gödel incompleteness and Turing uncomputability on physics, a question that has elicited varied responses [38, 39, 40, 41]. Recently, Heisenberg uncertainty has been related [42] to Gödel incompleteness through the notion of information theoretic incompleteness [43]. An interesting survey of the possible impact of Gödel's incompleteness theorems on physics may be found in Ref. [44].

We expect that the laws of physics possess sufficiently rich logical structure to express elementary arithmetic, as evidenced for example by the simple fact of existence of electronic computers. Hence a formalization of physics should be able to express metatheoretic propositions, such as provability in itself. If consistent, this formal system (and by extension, physics itself) will then be incomplete. Intuitively, we regard physical systems as ‘computing’ their own evolution [45], and thus believe that there is an algorithm to compute any property of a system. The Church-Turing thesis of computer science [22] may then be invoked to suggest that no physical process, interpreted as a procedure for computing a function, can be more powerful than Turing machines. Thus the incompleteness of physics could manifest in the physical uncomputability of Turing uncomputable problems [46].

On the other hand, a candidate theory of physics could be inconsistent through self-reference. It may not be straightforward to detect inconsistency in the laws of the theory. Gödel's second incompleteness theorem shows that the formal system P can prove its own consistency if and only if it is inconsistent [47]. One way to deal with the issue of consistency of a physical theory would be to axiomatize it and then try to demonstrate its consistency or inconsistency, failing which one may hope that it is consistent; or, one may try to prove its consistency metatheoretically. Even in the absence of a formalization of the theory, an inconsistency might be revealed through the prediction of some genuine paradox. The detection of an inconsistency would imply that the theory in question is inadequate, or, what is less likely, that Nature herself harbors an inconsistency [44]. Here we propose that the localization paradox of BHIP may signal an inconsistent self-reference in the semiclassical approach to black hole evolution.

To view the formal inconsistency of the BHIP localization problem as self-referential, it may be re-cast as a time paradox. Suppose Alice freefalls for a while, but before reaching the horizon, switches on a rocket and returns back to Bob. In this case, no paradox arises because Alice's and Bob's perspectives will coincide, and their respective observations will be continuously transformable into each other, within the semiclassical theory. However, if Alice chooses to continue to freefall, and enters the black hole at event ϵ , then Bob's perspective does not register ϵ , but instead the incompatible (possibly null) event η .

This situation may be described in the following somewhat fanciful language. The information about Alice (or M) propagates towards the black hole initially from an unambiguous past. If Alice does not fire her rocket but

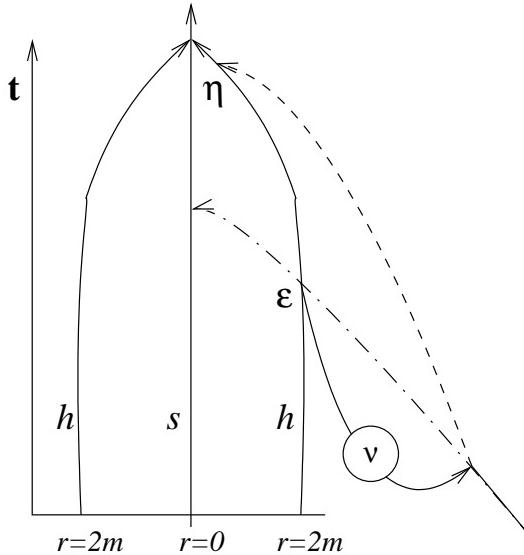


FIG. 1: Bird’s eye ‘meta-view’ of the BHIP localization paradox as a self-referential classical information circuit. Negative time travel is indicated through the curve labeled by the encircled “ v ”. (a) The lines s and h represent the singularity and the shrinking horizon, respectively. The dash-dotted and dashed curves are the worldlines of an infalling object, according to Alice’s and Bob’s perspectives, respectively. We note that η , if it happens, presumably occurs in the fully quantum gravity regime, and if it is a null event, the ‘ η -worldline’ will not intercept the horizon. The negative-time feed-back is triggered if and only if event ϵ happens. Prior to the feed-back induced nonlocal split, the object’s worldline exists in an unambiguous past.

freefalls into the black hole, a signal time-travels from her future self at the event ϵ to her past self occurring earlier, instructing the latter to shift to the incompatible worldline leading to η in Bob’s perspective. Thus, if an infalling body enters the black hole at ϵ , it ‘will not have entered’ the black hole during that event. And if η is a null event, the paradox is that if the body enters the black hole, then it will not have entered. In this sense, the BHIP localization problem may be viewed as a physical version of the liar’s paradox. If Alice fires her rocket to return back to Bob, no such time-traveling signal occurs. More generally, this signal is generated, and an infalling object ‘experiences’ a nonlocal splitting of the self into the two mutually incompatible perspectives, if and only if it crosses the horizon in Alice’s perspective. Since this time-traveling signal occurs across the perspectives, the self-reference is perceived, strictly speaking, in the ‘meta-perspective’ that has a bird’s eye view of both perspectives, rather than in the Alice perspective or Bob perspective alone (cf. Figure 1).

Although this self-reference is tied in a complicated way to the causal structure of spacetime in general relativity modified by quantum mechanics, the essential idea of the inconsistency as being due to a temporal self-reference, and of the incompleteness obtained by imposing consistency, can be roughly demonstrated using simple ‘self-referential circuits’ that compute a one-bit partial function (cf. Appendix A).

A quick way to impose consistency in the BHIP localization problem is to proscribe objects from falling into the inconsistent zone which is the evaporating black hole. Applied to any infalling body, this would suggest that the horizon never forms in finite time in the first place. Gödel incompleteness would then correspond to the situation that, if the theory is consistent, it could somehow not allow the dynamic formation of a solution (the Schwarzschild black hole) that it nevertheless allows to exist, because if this solution were formed in finite time, it would be inconsistent. One could then require that the detailed dynamics of the infalling matter, possibly involving new physics, would somehow conspire to prevent an object’s collapsing into the horizon. We will return to this point in detail in the next Section, where we consider this and various other proposals to resolve BHIP in this light.

III. TOWARDS RESOLVING BHIP VIA GÖDEL INCOMPLETENESS

Even if we admit that the semiclassical theory of black hole evaporation may be inconsistent in the above sense, nonrelativistic quantum mechanics and classical general relativity are arguably consistent in their own domains.

This suggests that the axioms of quantum mechanics are not compatible with those of general relativity, and that BHIP may be a manifestation of this incompatibility. Again, in the absence of a rigorous axiomatization of semiclassical general relativity, three broad operational responses to the situation may be considered in order to

eliminate the inconsistency, either by averting self-reference, or by invoking possibly new physics that would explain the Gödel-like incompleteness corresponding to an imposition of consistency: (A) to somehow thwart the full collapse of M into a black hole from happening in finite time; (B) to modify the semiclassical theory, with the modifications being understood as coming from the full theory of quantum gravity; (C) to modify standard non-relativistic quantum mechanics, and/or classical general relativity.

The first two options are considered sequentially in the following two subsections. Option (C) is considered in the next Section. Option (A) is concerned with the introduction of Gödel incompleteness in the form of prohibiting the formation of black holes in finite time. Option (B) admits the inconsistency in the initial phase of infall through the horizon, but enforces late-time consistency, by means of a black hole remnant or a black hole final state. Option (C) aims to modify one or both of the ingredient theories of semiclassical quantum gravity, i.e., quantum mechanics and general relativity, so that no BHIP-like self-reference occurs in putting them together.

A. Eternally collapsing objects instead of evaporating black holes.

As noted briefly at the end of Section II B, the simplest way of imposing consistency on the evolution of infalling bodies implies that non-zero mass black holes should never form in finite time. If we believe in the consistency of the semiclassical approach, we may then ‘predict’ the existence of a dynamical mechanism that would explain how a body may be prevented from collapsing into a black hole of non-zero mass in finite time. The physics of such a mechanism would supply the required Gödel incompleteness. It is possible that we would require new physics to fulfil this purpose, but we expect that there would be little departure from semiclassical theory near the horizon.

Remarkably, such a no-go mechanism may already exist in *classical* general relativity. In this scenario, what are conventionally considered to be black hole candidates are proposed to be eternally collapsing objects (ECOs) [49], with various initial mass distributions lead to $m = 0$ eventually (cf. Ref. [49] and references therein). Further, in Ref. [50], the inherently quantum functional Schrödinger formalism applied to the quantum collapse of and radiation from a 3+1 dimensional shell of matter finds that the event horizon may never form in finite time. In addition, the radiation as seen from the outside observer’s perspective turns out to be non-thermal [51], which allows for information to be emitted out.

There is some supporting observational evidence in terms of a quasar containing intrinsic magnetic moment [52], indicative of the absence of an event horizon.

Conventional belief in the existence of black holes rests on the exact Oppenheimer-Snyder solution to the general relativistic spherical collapse equations [53], in which the collapsing body is modelled as “pressureless dust”, and is shown to form a black hole in finite proper time that goes as $m^{-1/2}$. The point of departure to ECOs is to note that this solution, in which the collapsing fluid is implicitly of zero internal energy and zero heat flux, does not correspond to any physical fluid, whereas in a realistic situation, the collapsing fluid will have finite pressure and finite density gradient. In particular, radiation density and heat flux should increase sharply as the horizon is approached, and the build-up of gravitationally trapped radiation pressure is predicted to keep slowing down the collapse as the object becomes sufficiently compact. Accordingly, ECO theory predicts that massive objects suffering spherical gravitational collapse never actually form non-zero mass black holes. In evolving towards a black hole, an ECO burns its entire mass into radiation, so that the black hole that eventually forms at infinite proper time, has mass $m = 0$ [49]. Obviously, such a Schwarzschild black hole that lacks any entropy would resolve the BHIP information loss problem.

In situations where ECO’s are shown to be unavoidable, they could be a manifestation of Gödel incompleteness in the following sense: acceptance of ECOs leads us to the situation that, even though a Schwarzschild black hole occurs as a solution to the general relativity field equations, there are no initial conditions on M that can collapse into the black hole in finite time. The existence of such a dynamically unattainable solution can furnish a Gödel-like incompleteness corresponding to the theory’s consistency. This is analogous to the expressibility of an unprovable proposition in the formal system P , assumed to be consistent. ECO’s then may be a purely classical effect that anticipates the quantum effect of black hole evaporation, just as the classical black hole area theorem [54] anticipates the notion of black hole temperature.

To clarify the formal character of the incompleteness, we may regard the collection of physical bodies such as M as a formal system representing the deductive theory of general relativity (just as Turing machines or physical computers may serve as a formal system representing arithmetic). By direct physical evolution, this formal system ‘proves’ theorems of general relativity. More precisely, physical evolution drives a celestial body into various states, which can be interpreted metatheoretically as representing theorems in general relativity, just as a series of symbolic manipulations in P produces new formulas, which can be interpreted as theorems in arithmetic. We know ‘metatheoretically’, by direct mathematical insight, that a Schwarzschild solution of finite mass m exists. However, ECO theory implies that our formal system is constrained not to find this out in finite time. Gödel incompleteness would then correspond to the situation that the semiclassical Schwarzschild black hole is a true solution that cannot be detected by any

consistent formalization of the semiclassical theory of gravity.

A formalized proof of the existence of black holes would be an interpretation of the formation of a Schwarzschild black hole in finite time within the semiclassical theory, which would make the formalization inconsistent through BHIP. This is analogous to the situation that, according to Gödel's first incompleteness theorem, a proof of proposition (3) within P would make the formal system inconsistent through self-reference.

If the ECO scenario holds true generally, and is not restricted to isolated bodies that collapse with (approximately) spherical symmetry, this would offer further support for the view that the semiclassical approach is consistent but incomplete. The eternal-collapse scenario potentially provides the most conservative resolution of BHIP, since no new physics would be needed. If the BHIP localization problem can be shown to arise in the formation of any horizon, one could 'predict' ECO's as a generic consequence of the consistency of the semiclassical theory of gravity.

However, if it turns out that there are certain mass distributions for which the finite proper time formation of non-zero mass black holes cannot be avoided, we would be led to conclude that semiclassical gravity is probably inconsistent, and new physics, such as the possible alternatives discussed below, may have to be invoked in order to resolve BHIP.

B. Consistency through selection of an unambiguous future

Unlike the approach in the preceding subsection, the present one, which implements option (B), involves matter passing into the black hole, and thus the BHIP localization problem is unavoidable in the events pertaining to the initial phase of the infall through the horizon. However, one may restore consistency at late time events, by having the information localizations in the Alice and Bob versions somehow merge eventually. Although this does not avert the inconsistency in toto, we may be satisfied with restricting it to a finite measure and avoiding the prospect of 'eternal inconsistency', in which the two versions diverge forever.

There seems to be little freedom to alter the Alice version during the events of the initial phase of infall through the horizon of a large black hole, since (semi-)classical physics presumably holds at those events. Relatively speaking, there is some room to maneuver the Bob version, depending on the theory of quantum gravity. Accordingly, three broad scenarios of late-time resolution of the localization problem are available. First is: (a) that the event η occurs eventually after the breakdown of the semiclassical approximation, with both perspectives being agreed thereafter that the information is localized inside the black hole, which, in the Bob perspective, is now a black hole remnant or a naked singularity.

An alternative possibility is that the event η does not occur, as in the string theoretic description. This would have meant that the Alice and Bob perspectives remain 'eternally incompatible'. Therefore, one option is: (b) nonlocally transferring the information as seen in the Alice perspective to the Hawking radiation. As for Bob, he perceives the information of the infalling object somehow pass into the Hawking radiation without going through the horizon. Another option is: (c) eventually destroying the information in both Alice's and Bob's perspectives. This would make the localization problem redundant, but at the cost of unitarity.

Simple examples of the above three scenarios (a), (b) and (c) are presented in the following three headings in this Subsection, respectively. They had originally been proposed in connection with the BHIP information loss problem, rather than the localization problem. Here we point out that they can be adapted to address the latter at late-time events. In (b) and (c), we expect that this enforcement of consistency will produce Gödel-like incompleteness, as indeed confirmed below.

a. The information is localized in a naked singularity or black hole remnant.

In the first example, which illustrates scenario (a), during Hawking radiation the black hole retains all the initial matter together with negative energy quanta entangled with the Hawking radiation without mutual annihilation [4]. As the black hole's aggregate mass drops to zero in the semiclassical limit, its Hawking temperature ($\propto 1/m$) rises to infinity. It is further assumed that such a zero-mass, information-bearing object is somehow without detectable impact on low-energy experiments [2]. Crucially, it is argued that the horizon does not vanish but recedes to $r = 0$. To see this, consider the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (4)$$

where we have used natural units in which $G = c = 1$. At first sight, the metric when the mass m drops to zero seems to correspond to flat spacetime, endowed with Minkowski metric $ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$. However, a more careful

calculation shows

$$\lim_{r \mapsto 0^+} g_{00}(r = 2m) = 0; \quad \lim_{r \mapsto 0^+} g_{rr}(r = 2m) = \infty. \quad (5)$$

This corresponds to a kind of an ‘informationally dense’ naked singularity, with horizon at $r = 0$. Information about M exists in the full entangled state encompassing the singularity and the Hawking radiation. This provides a resolution to the information loss problem.

In regard to the localization problem, there is an initial incompatibility between the Alice and Bob perspectives because ϵ occurs at close to $r = 2m$, whereas η happens presumably at about $r = 0$, when the evaporating black hole’s size is of the order of Planck length. Thereafter an unambiguous future localization, and hence restoration of consistency, is assumed to occur, with unequivocal agreement between both perspectives that the information is localized at the quantum black hole singularity. A detailed quantum gravity treatment should presumably replace the singularity with a black hole remnant.

b. The information becomes localized in the Hawking radiation.

The second example, exemplifying scenario (b), is based on an interesting recent proposal to reconcile the unitarity of the black hole S-matrix with Hawking’s semiclassical arguments [6]. It aims to resolve the BHIP information loss problem by imposing a final-state boundary condition at the spacelike singularity inside the black hole, which causes the information inside it to be ‘ejected’ into the Hawking radiation. Here we will note that it also reconciles the perspectives of Alice and Bob, because the information is now unambiguously localized—outside the black hole.

The final state boundary condition imposed at the singularity requires the quantum state of $H_M \otimes H_{\text{in}}$ to be the maximally entangled state [7]

$${}_{M+\text{in}}\langle \Phi | (S \otimes I) \quad (6)$$

where ${}_{M+\text{in}}\langle \Phi | = N^{-1/2} \sum_{j=1}^N {}_M\langle j|_{\text{in}} \langle j|$, S is a unitary transformation, and $\{|j\rangle_M\}$ is an orthonormal basis for H_M . The effective transformation from H_M to H_{out} is seen to be [7]

$$T \equiv {}_{M+\text{in}}\langle \Phi | (S \otimes I) | \Phi \rangle_{\text{in+out}} = \frac{1}{N} S, \quad (7)$$

the effectively unitary, black hole S-matrix. The $1/N$ factor accounts for post-selection and indicates that a conventional measurement would have resulted in the final state (6) with probability $1/N$. This process may be viewed as a sort of quantum teleportation [55] that consumes the Unruh state entanglement in order to nonlocally transmit M ’s state to the outgoing Hawking radiation, but without the concomitant classical communication. This enables the two versions to agree on the late time localization of the information.

The application of this picture to a semiclassical theory of black hole evaporation in which η does not happen, could be as follows. For concreteness, we consider a string-theory-like description of black hole evaporation. In Bob’s perspective, the information about M does not pass through the horizon, but is somehow transferred into the Hawking radiation after being disrupted at the heated membrane. In Alice’s perspective, the information does initially pass through the horizon, after which the projection into the black hole final state ‘teleports’ the information into the Hawking radiation. Thus, an unambiguous future eventually appears, providing a late-time resolution to the localization problem. This imposition of consistency produces a Gödel-like incompleteness.

The event of projection of the state in $H_M \otimes H_{\text{in}}$ to ${}_{M+\text{in}}\langle \Phi | (S \otimes I)$ occurs after M enters into the horizon, as seen from Alice’s perspective. However, because in Bob’s perspective M does not enter the black hole, the scattering process characterized by T would have to be attributed in his perspective to some fundamentally indeterminate quantum gravity effect at the heated membrane, that destroys M , recreating it in H_{out} . We identify Gödel incompleteness with this inability of the theory to provide a detailed external-based description of the black hole S-matrix.

c. The information is destroyed.

The third example, a concrete instance of scenario (c), is based on a careful critique [7] of the above black hole final state proposal, where it is pointed out that departures from unitarity of T can arise due to interactions U between the collapsing body and the infalling part of the Hawking radiation. Because of the black hole final state, the resulting loss of information outside is not compensated for by any information available inside the black hole. This automatically reconciles the perspectives of Alice and Bob, because now that the information is destroyed, obviously both can assert the loss of information unparadoxically. This provides a late-time resolution to the BHIP localization problem but a negative resolution to the information loss problem.

According to this proposal, the effective modified transformation on the infalling body is, in place of Eq. (7),

$$T \equiv {}_{M+\text{in}}\langle \Phi | (S \otimes I) U | \Phi \rangle_{\text{in+out}} \equiv {}_{M+\text{in}}\langle \Phi | V | \Phi \rangle_{\text{in+out}}. \quad (8)$$

If and only if ${}_{M+\text{in}}\langle \Phi | V$ is a maximally entangled state is T unitary (after renormalization). If V is chosen to be a maximally (dis)entangling interaction, such as the controlled-sum gate $V|j, k\rangle = |j, (j+k) \bmod N\rangle$, then from Eq. (8), one has

$$T = \sum_n \frac{1}{N} |0\rangle \langle n|, \quad (9)$$

i.e., the state of the outgoing radiation is $|0\rangle_{\text{out}}$, irrespective of the incoming state $|m\rangle$ of the collapsing body. Interestingly, since the final state of the radiation is a fixed pure state $|0\rangle_{\text{out}}$, predictability is not lost. Such a nonunitary black hole evaporation would serve as a novel ‘quantum deletion’ mechanism [56].

As before, to apply this picture to a semiclassical theory of black hole evaporation in which η does not happen, we consider for concreteness a string-theory-like description of black hole evaporation. In Bob’s perspective, the information about M does not pass through the horizon, but is somehow disrupted and destroyed at the heated membrane. As a result, the Hawking radiation is a truly thermal mixture. In Alice’s perspective, the information passes through the horizon, and remains inside until the final state projection. The interaction U impairs the fidelity of the ‘teleportation’ of the inside information into the Hawking radiation, thereby destroying the information. Thus, an unambiguous future eventually appears, providing a late-time resolution to the localization problem.

For a reason similar to that in the preceding scenario (b), the imposition of consistency brings Gödel-like incompleteness corresponding to the fact that in Bob’s perspective the origin of the process represented by the operator T in Eq. (9) is indeterminate. This is because the actions of U and V , as well as the final state projection happen behind the horizon, a region inaccessible to the infalling object in his perspective.

IV. TOWARDS RESOLVING BHIP VIA AVOIDANCE OF SELF-REFERENCE

Finally, under option (C), we first consider the possibility that departure from standard quantum theory can help avoid the self-reference that leads to the BHIP localization problem. A naive way to do so would be to somehow ‘turn off’ Hawking radiation. This would prevent the evaporation of the black hole, and thus ensure that an *evaporating* black hole does not exist in the theory, with the event ϵ lying eternally to Bob’s future, as in classical general relativity. One problem with this approach is that it requires the new physics to apply at the horizon, where the semiclassical description of spacetime is expected to be reasonably valid. This may not be insurmountable, since the suppression of Hawking radiation would hardly be noticeable. Another problem is that of giving a covariant specification of a condition that forbids pair-production near the horizon, given that there is no (necessarily) strong curvature or other coordinate-invariant manifestation of the event horizon. A breakdown in covariance might be one price to pay.

A further possibility under option (C) is that standard general relativity is inaccurate in the classical domain. We will consider an extreme realization of this option. Since the relativity of spacetime is essential to the localization problem of BHIP, the paradox vanishes if space and time are not relativistic, but are absolute, somewhat in the sense of the philosopher Immanuel Kant [57]. Most physicists would probably consider this approach unwarranted, but given the seriousness of BHIP, we think it worth at least a passing mention.

In his theory of transcendental idealism, Kant maintained that time and space are pure intuitions and *a priori* forms of intuition. Considered from the empirical perspective, they form the absolute context to objects in experience, i.e., phenomenal objects open to scientific study. In this respect, his view of space and time is Newtonian. However, the absoluteness is epistemological rather than ontological in the sense that space and time are not objects of perception, and do not exist for objects in themselves. Considered from the transcendental perspective, space and time are pure, and exist subjectively as conditions of knowledge, i.e., as cognitive structuring imposed on perception by the mind [58]. Kant also maintained that the axioms of Euclidean geometry were *synthetic* and known *a priori*. The former qualification means that they are not true in any logically necessary way, and could be denied without contradiction, as in non-Euclidean geometries. Nevertheless the latter qualification indicates that knowledge of the axioms of geometry precedes our experience of objects, depending only on our pure intuition (imaginative visualization) of space and time. Strictly speaking, this description applies to perceptual space rather than physical space, but Kant may not have intended them to be different.

It turns out that the proposition of absolute space and time is not as difficult to implement as may at first seem. For example, it is known that the special theory of relativity can be apparently reproduced in Newtonian spacetime assuming Lorentz length contraction of metric rods and slowing down of clocks moving with respect to a putative

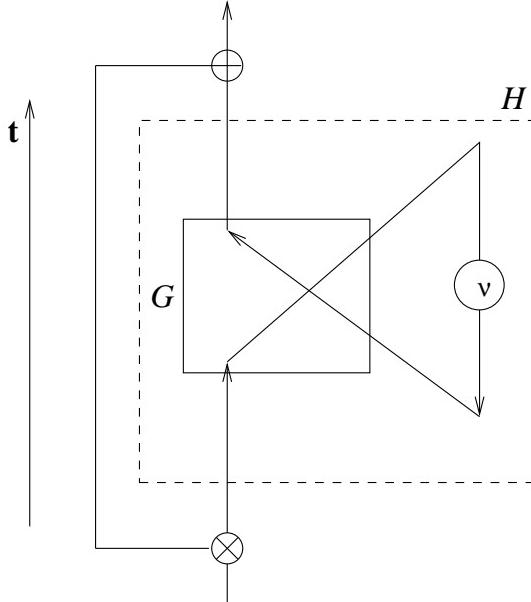


FIG. 2: Chronology violating subroutine H , with gate G , embedded in a circuit that forks the input, sending one copy through H , whose output is AND'ed with the other copy. The fork is indicated by the sign \otimes , and AND by \oplus . Negative time travel is indicated through the curve labeled by the encircled “ ν ” (Adapted from Figure 1a of Ref. [48]).

absolute rest frame. In a model of gravitation in absolute space and time, all events in spacetime, not merely causally connected ones, would be assumed to possess an absolute chronological ordering, with each event being assigned a unique spacetime point. Thus time paradoxes like the BHIP localization problem are automatically forbidden. For example, in Ref. [59], a detailed model of this kind has been proposed, in which black holes are replaced by stable frozen objects of the type discussed in Section III A. Not producing Hawking radiation, they do not evaporate, which eliminates BHIP.

V. CONCLUSIONS

Various attempts have been made to resolve BHIP, mostly aimed at understanding how information may be preserved during black hole evaporation. Here we focussed on the problem of information localization in BHIP, and argued that it signals an inconsistent self-reference in semiclassical gravity. This may be regarded as evidence of the inadequacy of the semiclassical treatment of black hole evolution, or that standard quantum mechanics and general relativity are incompatible. To restore consistency, we require to avert self-reference by modifying one or both of the latter theories, or introduce (new) physics that imparts to semiclassical gravity the incompleteness that would correspond to imposing consistency. Various scenarios have been discussed under this rubric.

APPENDIX A: INCOMPLETENESS DUE TO CONSISTENT SELF-REFERENCE

Syntactic Gödel incompleteness arising from imposing consistency in self-referential systems can be demonstrated using simple classical circuits. Self-reference can be introduced in a circuit by identifying two pieces of information with each other, i.e., by introducing a loop. Embedded in spacetime, such loops correspond to chronology violation-information flow through closed timelike lines [48]. Such ‘self-referential circuits’ allow an information carrier’s future self to interact with its past self. It is well known that this can lead to logical paradoxes of the following kind. One imagines travelling back in time and preventing the time-machine that permits us to travel back in time from being built. Consistency then demands that the permitted initial conditions should be compatible with self-referencing, in this case, time travel. This may be regarded as a kind of incompleteness.

As a simple toy illustration, we consider the chronology violating information circuit H , outlined by the dashed box in Figure 2. The encircled “ ν ” represents a negative time delay whereby a bit travels back in time. The two versions

of the bit passing through the gate G undergo the interaction [48]

$$G|x\rangle|y\rangle = |x \oplus y\rangle|y\rangle, \quad (\text{A1})$$

where the operation in the first ket is XOR (addition modulo 2). The first and second kets refer to the past and future selves of the bit, respectively. The past self is in the state $|x \oplus y\rangle$ on leaving G , and the future self is still in the state $|y\rangle$ in which it entered G . That the bit does not evolve outside the gate imposes the consistency condition [48]

$$x \oplus y = y. \quad (\text{A2})$$

In this case, the input value x to G must be zero, for no value of y satisfies the consistency condition (A2) if $x = 1$. This restriction due to the demand of consistency, on what inputs may be processed, may be regarded as the system's Gödel-like incompleteness. The condition (A2) does not uniquely fix the output from G , a problem that may be amended by adding supplementary conditions to the chronology violating bit. Alternatively, we can embed H as a subroutine in a larger circuit, as in Fig. 2, which is insensitive to this ambiguity. Here one evaluates the logical AND of a copy of the input bit with the output processed by G . The demand for consistency retrospectively prohibits an input $x = 1$ to the fork from the unambiguous past. However, an input $x = 0$ yields the unique output $y = 0$ at the unambiguous future. Thus the unary gate of Figure 2 implements the partial function ϕ on one bit, given by $\phi(0) = 0$ and $\phi(1)$ being undefined. Neither is the full gate universal nor does it require one (NAND or NOR).

This toy model of incompleteness is meaningful only if $x = 0$ and $x = 1$ are interpreted as strictly incompatible possibilities. If on the contrary, they are interpreted as basis states of a quantum observable (which are only classically incompatible), and G as the unitary, controlled-not operation $\sum_{x,y} |x \oplus y, y\rangle\langle x, y|$, one can avoid the restrictions on the input by allowing arbitrary mixed-state superpositions in the output $|y\rangle$ [48]. The reason is that the consistency condition is equivalent to the requirement that an evolutionary operator have a fixed point. Whereas in a quantum system, there is always such a fixed point (cf. below), in general, a classical system does not have such a fixed point.

The question of whether BHIP may find a solution in the space of superpositions of the ϵ - and η -worldlines must surely be answered in the negative, for the reason clarified in Section II. Likewise, the outputs $y = 0$ and $y = 1$ are indeed incompatible in the purely mathematical problem of evaluating a partial function over integers, represented by the circuit in Figure 2. Thus, our illustration of Gödel-like self-reference by a *classical* self-referential circuit is appropriate.

As a circuit version of the liar's paradox, we consider the gate G in Figure 2 given by [48]

$$G|x\rangle|y\rangle = |y \oplus 1\rangle|x\rangle. \quad (\text{A3})$$

The first and second kets correspond, respectively, to two statements X and Y . The action of the gate is to indicate the content of the statements, with X asserting the falsehood (NOT) of Y , and Y asserting the truth of X . This self-reference does not have the earlier temporal context. Eq. (A3) yields the consistency condition [48]

$$y \oplus 1 = y, \quad (\text{A4})$$

which is tantamount to requiring the NOT gate being equivalent to the identity operation. This cannot be satisfied, and hence rules out all inputs. (A quick way to see why a consistent quantum solution is not ruled out, is to note that the unitary version of G in Eq. (A3) acts as an identity operation between the ‘unambiguous past’ and ‘unambiguous future’ selves of the qubit, with the chronology violating qubit being described by any density operator $\rho = \frac{1}{2}[\hat{\mathbb{I}} + \zeta(|0\rangle\langle 1| + |1\rangle\langle 0|)]$ that is a fixed point of the NOT (bit-flip) operation, that is, $\rho = \hat{X}\rho$.)

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